

# GENERAL HEART CONSTRUCTION FOR TWIN TORSION PAIRS ON TRIANGULATED CATEGORIES

HIROYUKI NAKAOKA

**ABSTRACT.** In our previous article, we constructed an abelian category from any torsion pair on a triangulated category. This generalizes the heart of a  $t$ -structure and the ideal quotient by a cluster tilting subcategory. Recently, generalizing the quotient by a cluster tilting subcategory, Buan and Marsh showed that an integral preabelian category can be constructed as a quotient, from a rigid object in a triangulated category with some conditions. In this article, by considering a pair of torsion pairs, we make a simultaneous generalization of these two constructions.

## 1. INTRODUCTION AND PRELIMINARIES

For any category  $\mathcal{K}$ , we write abbreviately  $K \in \mathcal{K}$ , to indicate that  $K$  is an object of  $\mathcal{K}$ . For any  $K, L \in \mathcal{K}$ , let  $\mathcal{K}(K, L)$  denote the set of morphisms from  $K$  to  $L$ . If  $\mathcal{M}, \mathcal{N}$  are full subcategories of  $\mathcal{K}$ , then  $\mathcal{K}(\mathcal{M}, \mathcal{N}) = 0$  means that  $\mathcal{K}(M, N) = 0$  for any  $M \in \mathcal{M}$  and  $N \in \mathcal{N}$ . For each  $K \in \text{Ob}(\mathcal{K})$ , similarly  $\mathcal{K}(\mathcal{M}, K) = 0$  means that  $\mathcal{K}(M, K) = 0$  for any  $M \in \mathcal{M}$ . We denote the full subcategory of  $\mathcal{K}$  consisting of those  $K \in \mathcal{K}$  satisfying  $\mathcal{K}(\mathcal{M}, K) = 0$  by  $\mathcal{M}^\perp$ . Dually,  $\mathcal{K}(K, \mathcal{N}) = 0$  means  $\mathcal{K}(K, N) = 0$  for any  $N \in \mathcal{N}$ , and these form a full subcategory  ${}^\perp\mathcal{N} \subseteq \mathcal{K}$ . If  $\mathcal{K}$  is additive and  $\mathcal{N} \subseteq \mathcal{K}$  is a full additive subcategory, then  $\mathcal{K}/\mathcal{N}$  is defined to be the ideal quotient of  $\mathcal{K}$  by  $\mathcal{N}$ . Namely,  $\mathcal{K}/\mathcal{N}$  is an additive category defined by

- $\text{Ob}(\mathcal{K}/\mathcal{N}) = \text{Ob}(\mathcal{K})$ ,
- For any  $X, Y \in \mathcal{K}$ ,

$$(\mathcal{K}/\mathcal{N})(X, Y) = \mathcal{K}(X, Y) / \{f \in \mathcal{K}(X, Y) \mid f \text{ factors through some } N \in \mathcal{N}\}.$$

Throughout this article, we fix a triangulated category  $\mathcal{C}$ . Any subcategory of  $\mathcal{C}$  is a full, additive subcategory closed under isomorphisms and direct summands. For any object  $T \in \mathcal{C}$ ,  $\text{add}(T)$  denotes the full subcategory of  $\mathcal{C}$  consisting of direct summands of finite direct sums of  $T$ . When  $\mathcal{M}, \mathcal{N}$  are subcategories of  $\mathcal{C}$  and  $C \in \mathcal{C}$ , then the abbreviations  $\text{Ext}^1(\mathcal{M}, \mathcal{N}) = 0$  and  $\text{Ext}^1(\mathcal{M}, C) = 0$  and  $\text{Ext}^1(C, \mathcal{N}) = 0$  are defined similarly as above. For any pair of subcategories  $\mathcal{M}, \mathcal{N} \subseteq \mathcal{C}$ , we define  $\mathcal{M} * \mathcal{N} \subseteq \mathcal{C}$  as the full subcategory consisting of those  $C \in \mathcal{C}$  admitting some distinguished triangle

$$M \rightarrow C \rightarrow N \rightarrow M[1]$$

with  $M \in \mathcal{M}$  and  $N \in \mathcal{N}$ .

By definition [IY], a torsion pair  $(\mathcal{X}, \mathcal{Y})$  on  $\mathcal{C}$  is a pair of (full additive) subcategories  $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$  satisfying

- (i)  $\mathcal{C}(\mathcal{X}, \mathcal{Y}) = 0$ ,
- (ii)  $\mathcal{C} = \mathcal{X} * \mathcal{Y}$ .

Previously in [N], we showed that if we are given a torsion pair  $(\mathcal{X}, \mathcal{Y})$  on  $\mathcal{C}$ , then

$$(1.1) \quad ((\mathcal{X} * \mathcal{W}) \cap (\mathcal{W} * \mathcal{Y}[1])) / \mathcal{W}$$

becomes an abelian category, where  $\mathcal{W} = (\mathcal{X}[1] \cap \mathcal{Y})$ . This generalizes the following two constructions.

- (1) **The heart of a  $t$ -structure.** A  $t$ -structure is nothing other than a torsion pair  $(\mathcal{X}, \mathcal{Y})$  satisfying  $\mathcal{X}[1] \subseteq \mathcal{X}$ . In this case, (1.1) agrees with the heart [BBD].
- (2) **Ideal quotient by a cluster tilting subcategory.** A full additive subcategory  $\mathcal{T} \subseteq \mathcal{C}$  is a cluster tilting subcategory if and only if  $(\mathcal{T}[-1], \mathcal{T})$  is a torsion pair on  $\mathcal{C}$ . In this case, (1.1) agrees with the ideal quotient of  $\mathcal{C}$  by  $\mathcal{T}$ , which was shown to become abelian in [KZ] (originally in [BMR], or [KR] in 2-CY case).

Recently, generalizing the second case of a cluster tilting subcategory, Buan and Marsh showed that if  $T$  is a rigid (i.e.  $\text{Ext}^1(T, T) = 0$ ) object in a Hom-finite Krull-Schmidt triangulated category  $\mathcal{C}$  (over a field  $k$ ) with a Serre functor, then

$$(1.2) \quad \mathcal{C} / \mathcal{X}_T \quad (\text{where } \mathcal{X}_T = (\text{add}(T))^\perp)$$

becomes an integral preabelian category. (In the notation of [BM],  $\mathcal{X}_T$  is written as  $\mathcal{X}_T = (\text{add}(T))^\perp[1]$ . This is only due to the difference in the definition of  $\mathcal{M}^\perp$ . In [BM], for any  $\mathcal{M} \subseteq \mathcal{C}$ ,  $\mathcal{M}^\perp$  is defined to be the full subcategory of  $\mathcal{C}$  consisting of those  $C \in \mathcal{C}$  satisfying  $\text{Ext}^1(\mathcal{M}, C) = 0$ .)

As in [R] (and as quoted in [BM]), an additive category  $\mathcal{A}$  is *preabelian* if any morphism in  $\mathcal{A}$  has a kernel and a cokernel. A preabelian category is *left semi-abelian* if and only if for any pullback diagram

$$(1.3) \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta \downarrow & \square & \downarrow \gamma \\ C & \xrightarrow{\delta} & D \end{array}$$

in  $\mathcal{A}$ , “ $\delta$  is a cokernel morphism” implies “ $\alpha$  is epimorphic” [R]. A *right semi-abelian* category is characterized dually, using pushout diagrams. A *semi-abelian* category is defined to be a preabelian category which is both left semi-abelian and right semi-abelian.

A preabelian category  $\mathcal{A}$  is *left integral* if for any pullback diagram (1.3), “ $\delta$  is epimorphic” implies “ $\alpha$  is epimorphic”. A *right integral* category is defined dually, using pushout diagrams. An *integral* category is defined to be a preabelian category which is both left integral and right integral. Thus a preabelian category  $\mathcal{A}$  is semi-abelian whenever it is integral. Moreover if  $\mathcal{A}$  is integral, then the localization of  $\mathcal{A}$  by regular (= epimorphic and monomorphic) morphisms are shown to become abelian. In [BM], using this fact, Buan and Marsh made an abelian category out of their integral preabelian category  $\mathcal{C} / \mathcal{X}_T$ .

In this article, we generalize simultaneously Buan and Marsh’s  $\mathcal{C} / \mathcal{X}_T$  and our  $\underline{\mathcal{H}}$ , using a pair of torsion pairs. Starting from torsion pairs, we need no assumption on  $\mathcal{C}$ , except it is triangulated.

## 2. DEFINITION AND BASIC PROPERTIES

As before,  $\mathcal{C}$  is a fixed triangulated category. Any subcategory of  $\mathcal{C}$  is assumed to be full, additive, closed under isomorphisms and direct summands.

**Definition 2.1.** Let  $\mathcal{U}$  and  $\mathcal{V}$  be full additive subcategories of  $\mathcal{C}$ . We call  $(\mathcal{U}, \mathcal{V})$  a *cotorsion pair* if it satisfies

- (i)  $\text{Ext}^1(\mathcal{U}, \mathcal{V}) = 0$ ,
- (ii)  $\mathcal{C} = \mathcal{U} * \mathcal{V}[1]$ .

*Remark 2.2.*  $(\mathcal{U}, \mathcal{V})$  is a cotorsion pair if and only if  $(\mathcal{U}[-1], \mathcal{V})$  is a *torsion pair* in [IY]. (Unlike [BR], it does not require the closedness under shifts.) In this sense, a cotorsion pair is essentially the same as a torsion pair. However we prefer cotorsion pairs, for the sake of duality in shifts.

*Remark 2.3.* For any cotorsion pair  $(\mathcal{U}, \mathcal{V})$  on  $\mathcal{C}$ , we have  $\mathcal{U} = {}^\perp(\mathcal{V}[1])$  and  $\mathcal{V} = (\mathcal{U}[-1])^\perp$ .

Cotorsion pairs generalize  $t$ -structures and cluster tilting subcategories, as follows.

**Example 2.4.** (cf. Definition 2.6 in [ZZ], Proposition 2.6 in [N])

- (1) A *t-structure* is a pair of subcategories  $(\mathcal{X}, \mathcal{Y})$  where  $(\mathcal{U}, \mathcal{V}) = (\mathcal{X}[1], \mathcal{Y})$  is a cotorsion pair satisfying  $\mathcal{U}[1] \subseteq \mathcal{U}$ . This is also equivalent to  $\mathcal{V}[-1] \subseteq \mathcal{V}$ .
- (2) A *co-t-structure* is a pair of subcategories  $(\mathcal{X}, \mathcal{Y})$  where  $(\mathcal{U}, \mathcal{V}) = (\mathcal{X}[1], \mathcal{Y})$  is a cotorsion pair satisfying  $\mathcal{U}[-1] \subseteq \mathcal{U}$ . This is also equivalent to  $\mathcal{V}[1] \subseteq \mathcal{V}$ .
- (3) A cotorsion pair  $(\mathcal{U}, \mathcal{V})$  is called *rigid* if  $\text{Ext}^1(\mathcal{U}, \mathcal{U}) = 0$ . This is also equivalent to  $\mathcal{U} \subseteq \mathcal{V}$ .
- (4) A subcategory  $\mathcal{T} \subseteq \mathcal{C}$  is a *cluster tilting* subcategory if and only if  $(\mathcal{T}, \mathcal{T})$  is a cotorsion pair.

*Remark 2.5.* Using a result in [AN], we can characterize a co- $t$ -structure by the vanishing of an abelian category  $\underline{\mathcal{H}}$  defined as below. In fact, a cotorsion pair  $(\mathcal{U}, \mathcal{V})$  becomes a co- $t$ -structure if and only if it satisfies  $\underline{\mathcal{H}} = 0$ .

In [N], we showed the following.

**Theorem 2.6.** (Theorem 6.4 in [N]) Let  $(\mathcal{U}, \mathcal{V})$  be a cotorsion pair on  $\mathcal{C}$ . If we define full subcategories of  $\mathcal{C}$  by

$$\mathcal{W} = \mathcal{U} \cap \mathcal{V}, \quad \mathcal{C}^- = \mathcal{U}[-1] * \mathcal{W}, \quad \mathcal{C}^+ = \mathcal{W} * \mathcal{V}[1], \quad \mathcal{H} = \mathcal{C}^+ \cap \mathcal{C}^-,$$

then the ideal quotient  $\mathcal{H}/\mathcal{W}$  becomes abelian.

In this article, we generalize this to the case of pairs of cotorsion pairs. We work on a pair of cotorsion pairs  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  on  $\mathcal{C}$  satisfying  $\text{Ext}^1(\mathcal{S}, \mathcal{V}) = 0$ . Since a “pair of pairs” is a bit confusing, we use the following terminology.

**Definition 2.7.** A pair of cotorsion pairs  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  on  $\mathcal{C}$  is called a *twin cotorsion pair* if it satisfies

$$(2.1) \quad \text{Ext}^1(\mathcal{S}, \mathcal{V}) = 0.$$

Remark that this condition is equal to  $\mathcal{S} \subseteq \mathcal{U}$ , and also to  $\mathcal{T} \supseteq \mathcal{V}$ .

**Definition 2.8.** Let  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  be a twin cotorsion pair on  $\mathcal{C}$ . We define subcategories of  $\mathcal{C}$  by

$$\mathcal{W} = \mathcal{T} \cap \mathcal{U}, \quad \mathcal{C}^- = \mathcal{S}[-1] * \mathcal{W}, \quad \mathcal{C}^+ = \mathcal{W} * \mathcal{V}[1], \quad \mathcal{H} = \mathcal{C}^+ \cap \mathcal{C}^-.$$

Each of  $\mathcal{C}, \mathcal{C}^+, \mathcal{C}^-, \mathcal{H}$  contains  $\mathcal{W}$  as a subcategory. We denote their ideal quotients by  $\mathcal{W}$  by  $\underline{\mathcal{C}}, \underline{\mathcal{C}}^+, \underline{\mathcal{C}}^-, \underline{\mathcal{H}}$ . Thus we obtain a sequence of full additive subcategories

$$\underline{\mathcal{H}} \subseteq \underline{\mathcal{C}}^+ \subseteq \underline{\mathcal{C}}, \quad \underline{\mathcal{H}} \subseteq \underline{\mathcal{C}}^- \subseteq \underline{\mathcal{C}}.$$

For any morphism  $f \in \mathcal{C}(A, B)$ , we denote its image in  $\underline{\mathcal{C}}(A, B)$  by  $\underline{f}$ .

*Remark 2.9.* Since  $\mathcal{W}$  is closed under direct summands, for any  $C \in \mathcal{C}$  we have

$$C = 0 \text{ in } \underline{\mathcal{C}} \iff C \in \mathcal{W}.$$

**Example 2.10.**

- (1) A single cotorsion pair can be regarded as a degenerated case of a twin cotorsion pair. A twin cotorsion pair  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  is a single cotorsion pair (namely,  $(\mathcal{S}, \mathcal{T}) = (\mathcal{U}, \mathcal{V})$ ) if and only if  $\mathcal{S} = \mathcal{U}$  if and only if  $\mathcal{T} = \mathcal{V}$ . In this case, (since  $\mathcal{W}, \mathcal{C}^+, \mathcal{C}^-$  and  $\mathcal{H}$  agrees with those in Theorem 2.6,)  $\underline{\mathcal{H}}$  becomes abelian as in Theorem 2.6.
- (2) Another extremal case is when  $\mathcal{T} = \mathcal{U}$ . Remark that in this case,  $\mathcal{S} \subseteq \mathcal{T}$  and  $\mathcal{U} \supseteq \mathcal{V}$  holds. In particular,  $(\mathcal{S}, \mathcal{T})$  is rigid.

As shown in [BM], if  $T$  is a rigid object in a Hom-finite Krull-Schmidt triangulated category  $\mathcal{C}$  (over a field  $k$ ) with a Serre functor, then  $(\mathcal{S}, \mathcal{T}) = (\text{add}(T)[1], \mathcal{X}_T)$  and  $(\mathcal{U}, \mathcal{V}) = (\mathcal{X}_T, (\mathcal{X}_T)^\perp[-1])$  are cotorsion pairs (Lemma 1.2 in [BM]). Since  $T$  is rigid, this pair satisfies  $\text{add}(T)[1] \subseteq \mathcal{X}_T$ . Thus  $\text{Ext}^1(\text{add}(T)[1], (\mathcal{X}_T)^\perp[-1]) = 0$ , and  $(\text{add}(T)[1], \mathcal{X}_T), (\mathcal{X}_T, (\mathcal{X}_T)^\perp[-1])$  becomes a twin cotorsion pair.

In this case, we have  $\underline{\mathcal{H}} = \mathcal{C}/\mathcal{X}_T$ , and it was shown in [BM] that this category becomes integral preabelian. (Remark that when  $\mathcal{T} = \mathcal{U}$ , generally we have  $\mathcal{W} = \mathcal{T} = \mathcal{U}$  and  $\mathcal{C}^+ = \mathcal{C}^- = \mathcal{C}$ .)

*Remark 2.11.* A similar situation to (2) in Example 2.10 appears in [BR] as a TTF-triple. A *TTF-triple* on  $\mathcal{C}$  is a triplet  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  of subcategories of  $\mathcal{C}$ , in which both  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{Y}, \mathcal{Z})$  are  $t$ -structures.

*Lemma 2.12.*

- (1) If  $U[-1] \rightarrow A \xrightarrow{f} B \rightarrow U$  is a distinguished triangle in  $\mathcal{C}$  satisfying  $U \in \mathcal{U}$ , then  $A \in \mathcal{C}^-$  implies  $B \in \mathcal{C}^-$ .
- (2) If  $S[-1] \rightarrow A \xrightarrow{f} B \rightarrow S$  is a distinguished triangle in  $\mathcal{C}$  satisfying  $S \in \mathcal{S}$ , then  $B \in \mathcal{C}^-$  implies  $A \in \mathcal{C}^-$ .

*Proof.* (1) Take distinguished triangles

$$\begin{aligned} S_A[-1] &\xrightarrow{s_A} A \xrightarrow{w_A} W_A \rightarrow S_A & (S_A \in \mathcal{S}, W_A \in \mathcal{W}), \\ S_B[-1] &\xrightarrow{s_B} B \xrightarrow{t_B} T_B \rightarrow S_B & (S_B \in \mathcal{S}, T_B \in \mathcal{T}). \end{aligned}$$

Since  $\text{Ext}^1(S_A, T_B) = 0$ ,  $f$  induces a morphism of triangles

$$\begin{array}{ccccccc} S_A[-1] & \xrightarrow{s_A} & A & \xrightarrow{w_A} & W_A & \longrightarrow & S_A \\ \downarrow & \circlearrowleft & \downarrow f & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ S_B[-1] & \xrightarrow{s_B} & B & \xrightarrow{t_B} & T_B & \longrightarrow & S_B. \end{array}$$

It suffices to show  $T_B \in \mathcal{U}$ .

Take any  $V^\dagger \in \mathcal{V}$ , and any  $v \in \mathcal{C}(T_B, V^\dagger[1])$ . Since  $\text{Ext}^1(W_A, V^\dagger) = 0$ , we have  $v \circ t_B \circ f = 0$ . Applying this to the given distinguished triangle, we see that  $v \circ t_B$  factors through  $U$ ,

$$\begin{array}{ccccc} U[-1] & \rightarrow & A & \xrightarrow{f} & B & \rightarrow & U \\ & & & & \downarrow t_B & \searrow \circ & \\ & & & & T_B & & \\ & & & & \downarrow v & \swarrow & \\ & & & & V^\dagger[1] & & \end{array}$$

and  $v \circ t_B = 0$  follows from  $\text{Ext}^1(U, V^\dagger) = 0$ . Thus  $v$  factors through  $S_B$ ,

$$\begin{array}{ccccc} S_B[-1] & \rightarrow & B & \xrightarrow{t_B} & T_B & \rightarrow & S_B \\ & & & & \downarrow v & \searrow \circ & \\ & & & & V^\dagger[1] & & \end{array}$$

which means  $v = 0$ , since  $\text{Ext}^1(S_B, V^\dagger) = 0$ .

(2) Take distinguished triangles

$$\begin{aligned} S_B[-1] &\xrightarrow{s_B} B \xrightarrow{w_B} W_B \rightarrow S_B \quad (S_B \in \mathcal{S}, W_B \in \mathcal{W}), \\ X &\rightarrow A \xrightarrow{w_B \circ f} W_B \rightarrow X[1]. \end{aligned}$$

Then by the octahedral axiom,

$$S[-1] \rightarrow X \rightarrow S_B[-1] \rightarrow S$$

is also a distinguished triangle. This implies  $X \in \mathcal{S}[-1]$ , and thus  $A \in \mathcal{C}^-$ .

$$\begin{array}{ccccc} S[-1] & \rightarrow & X & \rightarrow & S_B[-1] \\ & \searrow \circ & \downarrow & \searrow \circ & \\ & & A & \xrightarrow{f} & B \\ & \searrow & \downarrow & \swarrow w_B & \\ & & W_B & & \end{array}$$

□

Dually, the following holds.

*Lemma 2.13.*

- (1) If  $T \rightarrow A \xrightarrow{f} B \rightarrow T[1]$  is a distinguished triangle in  $\mathcal{C}$  satisfying  $T \in \mathcal{T}$ , then  $B \in \mathcal{C}^+$  implies  $A \in \mathcal{C}^+$ .
- (2) If  $V \rightarrow A \xrightarrow{f} B \rightarrow V[1]$  is a distinguished triangle in  $\mathcal{C}$  satisfying  $V \in \mathcal{V}$ , then  $A \in \mathcal{C}^+$  implies  $B \in \mathcal{C}^+$ .

The following Lemma is trivial.

*Lemma 2.14.* Let  $S_X[-1] \xrightarrow{s_X} X \xrightarrow{t_X} T_X \rightarrow S_X$  be a distinguished triangle, with  $S_X \in \mathcal{S}$  and  $T_X \in \mathcal{T}$ . If a morphism  $x \in \mathcal{C}(X, Y)$  factors through some  $T \in \mathcal{T}$ , then  $x$  factors through  $T_X$ .

Similar statement also holds for a distinguished triangle

$$U_X[-1] \xrightarrow{u_X} X \xrightarrow{v_X} U_X \rightarrow V_X \quad (U_X \in \mathcal{U}, V_X \in \mathcal{V})$$

and a morphism  $x \in \mathcal{C}(X, Y)$  factoring some  $V \in \mathcal{V}$ .

*Proof.* If  $x$  factors through  $T \in \mathcal{T}$ , then  $x \circ s_X = 0$  follows from  $\text{Ext}^1(S_X, T) = 0$ . Thus it factors through  $t_X$ . Similarly for the latter part.  $\square$

*Remark 2.15.* The dual of Lemma 2.14 also holds.

### 3. ADJOINTS

In the following, we fix a twin cotorsion pair  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  on  $\mathcal{C}$ . Since this assumption is self-dual, we often omit proofs of dual statements.

**Definition 3.1.** For any  $C \in \mathcal{C}$ , define  $K_C \in \mathcal{C}$  and  $k_C \in \mathcal{C}(K_C, C)$  as follows:

1. Take a distinguished triangle

$$S[-1] \rightarrow C \xrightarrow{a} T \rightarrow S \quad (S \in \mathcal{S}, T \in \mathcal{T})$$

2. then, take a distinguished triangle

$$U \rightarrow T \xrightarrow{b} V[1] \rightarrow U[1] \quad (U \in \mathcal{U}, V \in \mathcal{V})$$

3. and then, take a distinguished triangle

$$V \rightarrow K_C \xrightarrow{k_C} C \xrightarrow{b \circ a} V[1].$$

By the octahedral axiom,  $S[-1] \rightarrow K_C \xrightarrow{k_C} C \rightarrow S$  is also a distinguished triangle.

$$\begin{array}{ccccc}
 S[-1] & \xrightarrow{\quad} & K_C & \xrightarrow{\quad} & U \\
 & \searrow & \downarrow k_C & \searrow & \downarrow \\
 & & C & \xrightarrow{a} & T \\
 & \searrow & \downarrow & \searrow & \downarrow \\
 & & V[1] & & 
 \end{array}$$

**Claim 3.2.**

- (1) For any  $C \in \mathcal{C}$ , we have  $K_C \in \mathcal{C}^-$ .
- (2) If  $C \in \mathcal{C}^+$ , then  $K_C \in \mathcal{H}$ .

*Proof.* We use the notation in Definition 3.1.

- (1) Remark that we have a distinguished triangle

$$V \rightarrow U \rightarrow T \xrightarrow{b} V[1].$$

Since  $\mathcal{T} \supseteq \mathcal{V} \ni V$  and  $\mathcal{T}$  is closed under extensions, it follows  $U \in \mathcal{U} \cap \mathcal{T} = \mathcal{W}$ . Since  $S[-1] \rightarrow K_C \rightarrow U \rightarrow S$  is a distinguished triangle, we obtain  $K_C \in \mathcal{C}^-$ .

- (2) Since  $V \rightarrow K_C \rightarrow C \rightarrow V[1]$  is a distinguished triangle, this immediately follows from Lemma 2.13.  $\square$

Dually, we define as follows.

**Definition 3.3.** For any  $C \in \mathcal{C}$ , define  $Z_C \in \mathcal{C}$  and  $z_C \in \mathcal{C}(C, Z_C)$  as follows:

1. Take a distinguished triangle

$$V \rightarrow U \xrightarrow{a} C \rightarrow V[1] \quad (U \in \mathcal{U}, V \in \mathcal{V})$$

2. then, take a distinguished triangle

$$T[-1] \rightarrow S[-1] \xrightarrow{b} U \rightarrow T \quad (S \in \mathcal{S}, T \in \mathcal{T})$$

3. and then, take a distinguished triangle

$$S[-1] \xrightarrow{a \circ b} C \xrightarrow{z_C} Z_C \rightarrow S.$$

By the octahedral axiom,  $V \rightarrow T \rightarrow Z_C \rightarrow V[1]$  is also a distinguished triangle.

**Claim 3.4.**

- (1) For any  $C \in \mathcal{C}$ , we have  $Z_C \in \mathcal{C}^+$ .
- (2) If  $C \in \mathcal{C}^-$ , then  $Z_C \in \mathcal{H}$ .

*Proof.* This is the dual of Claim 3.4. □

**Proposition 3.5.** For any  $C \in \mathcal{C}$ , let  $K_C \xrightarrow{k_C} C$  be as in Definition 3.1. Then for any  $X \in \mathcal{C}^-$ ,

$$\underline{k}_C \circ -: \underline{\mathcal{C}}^-(X, K_C) \rightarrow \underline{\mathcal{C}}(X, C)$$

is bijective.

*Proof.* Take a distinguished triangle

$$S_X[-1] \xrightarrow{s_X} X \xrightarrow{w_X} W_X \rightarrow S_X \quad (S_X \in \mathcal{S}, W_X \in \mathcal{W}).$$

First we show the injectivity. Suppose  $x \in \mathcal{C}(X, K_C)$  satisfies  $\underline{k}_C \circ x = 0$ . By definition, this means that  $k_C \circ x$  factors through some object in  $\mathcal{W}$ . Thus by Lemma 2.14,  $k_C \circ x$  factors through  $w_X$ . This gives a morphism of triangles

$$\begin{array}{ccccccc} S_X[-1] & \xrightarrow{s_X} & X & \xrightarrow{w_X} & W_X & \rightarrow & S_X \\ \downarrow & \circlearrowleft & \downarrow x & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ V & \longrightarrow & K_C & \xrightarrow{k_C} & C & \rightarrow & V[1]. \end{array}$$

Since  $\text{Ext}^1(S_X, V) = 0$ , it follows  $x \circ S_X = 0$ , and thus  $x$  factors through  $W_X$ ,

$$\begin{array}{ccccc} S_X[-1] & \xrightarrow{s_X} & X & \xrightarrow{w_X} & W_X \\ & \searrow & \downarrow x & \searrow & \downarrow \\ & & K_C & & \end{array}$$

0

which means  $\underline{x} = 0$ .

Second, we show the surjectivity. Take any  $y \in \mathcal{C}(X, C)$ . In the notation of Definition 3.1,

$$\begin{array}{ccccc}
 S[-1] & \xrightarrow{\quad} & K_C & \xrightarrow{\quad} & U \\
 & \searrow \circlearrowleft & \downarrow k_C \circlearrowleft & & \swarrow \circlearrowleft \\
 & & C & \xrightarrow{a} & T \\
 & & \downarrow b \circ a \circlearrowleft & & \swarrow \circlearrowleft \\
 & & V[1] & & 
 \end{array}$$

since  $\text{Ext}^1(S_X, T) = 0$ ,  $y$  induces a morphism of triangles

$$\begin{array}{ccccccc}
 S_X[-1] & \xrightarrow{s_X} & X & \xrightarrow{w_X} & W_X & \rightarrow & S_X \\
 \downarrow \circlearrowleft & & \downarrow y \circlearrowleft & & \downarrow \exists_t \circlearrowleft & & \downarrow \circlearrowleft \\
 S[-1] & \rightarrow & C & \xrightarrow{a} & T & \rightarrow & S.
 \end{array}$$

Since  $\text{Ext}^1(W_X, V) = 0$ , we obtain

$$b \circ a \circ y = b \circ t \circ w_X = 0 \circ w_X = 0.$$

Thus  $y$  factors through  $k_C$ .

$$\begin{array}{ccccc}
 & X & & & \\
 & \swarrow & \searrow & \searrow 0 & \\
 V & \rightarrow & K_C & \xrightarrow{k_C} & C & \xrightarrow{b \circ a} & V[1] \\
 & & \downarrow \circlearrowleft & & \downarrow \circlearrowleft & & 
 \end{array}$$

□

Dually, we have the following.

**Proposition 3.6.** *For any  $C \in \mathcal{C}$ , let  $C \xrightarrow{z_C} Z_C$  be as in Definition 3.3. Then, for any  $Y \in \mathcal{C}^+$ ,*

$$- \circ z_C: \underline{\mathcal{C}}^+(Z_C, Y) \rightarrow \underline{\mathcal{C}}(C, Y)$$

*is bijective.*

In the terminology of [B], Proposition 3.6 means that for any  $C \in \mathcal{C}$ ,  $Z_C \xrightarrow{z_C} C$  gives a reflection of  $C$  along the inclusion functor  $\underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$ . As a corollary, we obtain the following.

**Corollary 3.7.** *(Proposition 3.1.2 and Proposition 3.1.3 in [B]) The inclusion functor  $i^+: \underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$  has a left adjoint  $\tau^+: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}^+$ . If we denote the adjunction by  $\eta: \text{Id}_{\underline{\mathcal{C}}} \Rightarrow i^+ \circ \tau^+$ , then there exists a natural isomorphism  $Z_C \cong \tau^+(C)$  in  $\underline{\mathcal{C}}$ , compatible with  $z_C$  and  $\eta_C$ .*

$$\begin{array}{ccc}
 C & \xrightarrow{z_C} & Z_C \\
 \eta_C \searrow & \circlearrowleft & \swarrow \cong \\
 & \tau^+(C) & 
 \end{array}$$

*In particular,  $Z_C$  is determined up to an isomorphism in  $\underline{\mathcal{C}}$ .*



Dually the following holds.

**Corollary 3.8.** *The inclusion functor  $i^- : \underline{\mathcal{C}}^- \hookrightarrow \underline{\mathcal{C}}$  has a right adjoint  $\tau^- : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}^-$ . For any  $C \in \mathcal{C}$ , there is a natural isomorphism  $K_C \cong \tau^-(C)$  in  $\underline{\mathcal{C}}^-$ .*

**Lemma 3.9.** *For any  $C \in \mathcal{C}$ , the following are equivalent.*

- (1)  $\tau^+(C) = 0$ .
- (2)  $Z_C \in \mathcal{W}$
- (3)  $C \in \mathcal{U}$ .

*Proof.* The equivalence between (1) and (2) follows from Remark 2.9 and Corollary 3.7.

Suppose (2) holds. Then, in the notation of the following diagram,

$$(3.1) \quad \begin{array}{ccccc} & & S[-1] & & \\ & & \downarrow b & \searrow & \\ & & U & \xrightarrow{a} & C & \xrightarrow{c} & V[1] \\ & & & \searrow z_C & \nearrow d & \\ & & & & Z_C & \\ & & & \nearrow e & \\ & & T & & \end{array}$$

we have  $d = 0$  since  $\text{Ext}^1(Z_C, V) = 0$ . Thus it follows  $c = 0$  and  $C$  becomes a direct summand of  $U$ , which means  $C \in \mathcal{U}$ .

Conversely, suppose (3) holds. Then in the notation in (3.1), we may assume  $a$  and  $e$  are isomorphisms. Since  $T \in \mathcal{W}$ , we obtain  $Z_C \in \mathcal{W}$ .  $\square$

Dually we have the following.

**Lemma 3.10.** *For any  $C \in \mathcal{C}$ , the following are equivalent.*

- (1)  $\tau^-(C) = 0$ .
- (2)  $K_C \in \mathcal{W}$
- (3)  $C \in \mathcal{T}$ .

By the same argument as in [AN], we can also show the following. Since we do not use it in this article, we only introduce the results.

**Proposition 3.11.** *The inclusion functor  $i_{\mathcal{U}} : \underline{\mathcal{U}} = \mathcal{U}/\mathcal{W} \hookrightarrow \underline{\mathcal{C}}$  has a right adjoint  $\sigma_{\mathcal{U}} : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{U}}$ . For any  $C \in \mathcal{C}$ ,  $\sigma_{\mathcal{U}}(C)$  is naturally isomorphic to  $U_C \in \underline{\mathcal{U}}$  appearing in a distinguished triangle*

$$V_C \rightarrow U_C \xrightarrow{u_C} C \xrightarrow{v_C} V_C[1] \quad (U_C \in \mathcal{U}, V_C \in \mathcal{V}).$$

Moreover for any  $C \in \mathcal{C}$ , the following are equivalent.

- (1)  $\sigma_{\mathcal{U}}(C) = 0$ .
- (2)  $U_C \in \mathcal{W}$ .
- (3)  $C \in \mathcal{C}^+$ .

Dually, the inclusion functor  $i_{\mathcal{T}} : \underline{\mathcal{T}} = \mathcal{T}/\mathcal{W} \hookrightarrow \underline{\mathcal{C}}$  admits a left adjoint  $\sigma_{\mathcal{T}} : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{T}}$ . We have a natural isomorphism  $\sigma_{\mathcal{T}}(C) \cong T_C$  in  $\underline{\mathcal{C}}$  for any  $C \in \mathcal{C}$ , and

$$\sigma_{\mathcal{T}}(C) = 0 \iff T_C \in \mathcal{W} \iff C \in \mathcal{C}^-,$$

where  $S_C[-1] \rightarrow C \rightarrow T_C \rightarrow S_C$  is a distinguished triangle with  $S_C \in \mathcal{S}, T_C \in \mathcal{T}$ .

4.  $\underline{\mathcal{H}}$  IS PREABELIAN

Now we construct the (co-)kernel of a morphism in  $\underline{\mathcal{H}}$ .

**Definition 4.1.** For any  $A \in \mathcal{C}^-$ ,  $B \in \mathcal{C}$  and  $f \in \mathcal{C}(A, B)$ , define  $M_f \in \mathcal{C}$  and  $m_f \in \mathcal{C}(B, M_f)$  as follows.

1. Take a distinguished triangle

$$S_A[-1] \xrightarrow{s_A} A \xrightarrow{w_A} W_A \rightarrow S_A,$$

2. then, take a distinguished triangle

$$S_A[-1] \xrightarrow{f \circ s_A} B \xrightarrow{m_f} M_f \rightarrow S_A.$$

$$\begin{array}{ccccc} S_A[-1] & & & & \\ s_A \downarrow & \searrow & & & \\ A & \xrightarrow{f} & B & \xrightarrow{m_f} & M_f \\ w_A \downarrow & & & & \\ W_A & & & & \end{array}$$

**Proposition 4.2.** For any  $A \in \mathcal{C}^-$ ,  $B \in \mathcal{C}$  and  $f \in \mathcal{C}(A, B)$ , let  $B \xrightarrow{m_f} M_f$  be as in Definition 4.1. Then, we have the following.

- (1)  $\underline{m}_f \circ \underline{f} = 0$ .
- (2)  $m_f$  induces a bijection

$$- \circ \underline{m}_f : \underline{\mathcal{C}}(M_f, Y) \xrightarrow{\cong} \{\beta \in \underline{\mathcal{C}}(B, Y) \mid \beta \circ \underline{f} = 0\}$$

for any  $Y \in \mathcal{C}^+$ .

- (3) If  $B \in \mathcal{C}^-$ , then  $M_f \in \mathcal{C}^-$ .

*Proof.* (1) is trivial, since there is a morphism of triangles

$$\begin{array}{ccccccc} S_A[-1] & \xrightarrow{s_A} & A & \xrightarrow{w_A} & W_A & \rightarrow & S_A \\ \parallel & & \circ & f \downarrow & \circ & \downarrow & \circ \\ S_A[-1] & \rightarrow & B & \xrightarrow{m_f} & M_f & \rightarrow & S_A. \end{array}$$

(3) follows from Lemma 2.12.

We show (2). Take a distinguished triangle

$$V_Y \rightarrow W_Y \xrightarrow{w_Y} Y \xrightarrow{v_Y} V_Y[1] \quad (V_Y \in \mathcal{V}, W_Y \in \mathcal{W}).$$

To show the injectivity, suppose  $x \in \mathcal{C}(M_f, Y)$  satisfies  $x \circ \underline{m}_f = 0$ . By definition  $x \circ m_f$  factors through some object in  $\mathcal{W}$ . By (the dual of) Lemma 2.14,  $x \circ m_f$  factors through  $w_Y$ , and thus we obtain a morphism of triangles

$$\begin{array}{ccccccc} S_A[-1] & \xrightarrow{f \circ s_A} & B & \xrightarrow{m_f} & M_f & \rightarrow & S_A \\ \downarrow & & \downarrow & \circ & \downarrow & x & \downarrow \\ V_Y & \rightarrow & W_Y & \xrightarrow{w_Y} & Y & \xrightarrow{v_Y} & V_Y[1]. \end{array}$$

Since  $\text{Ext}^1(S_A, V_Y) = 0$ ,  $x$  factors through  $W_Y$ , which means  $\underline{x} = 0$ .

To show the surjectivity, suppose  $y \in \mathcal{C}(B, Y)$  satisfies  $\underline{y} \circ \underline{f} = 0$ . By the same argument as above, we see that  $y \circ f$  factors  $W_Y$ . This implies  $y \circ f \circ s_A = 0$ , since  $\text{Ext}^1(S_A, W_Y) = 0$ . Thus  $y$  factors  $m_f$ .

$$\begin{array}{ccccc} S_A[-1] & \xrightarrow{f \circ s_A} & B & \xrightarrow{m_f} & M_f \rightarrow S_A \\ & \searrow \circlearrowleft & \downarrow y & \searrow \circlearrowleft & \\ & & Y & & \end{array}$$

$0$

□

Dually, we have the following:

*Remark 4.3.* For any  $A \in \mathcal{C}$ ,  $B \in \mathcal{C}^+$  and any  $f \in \mathcal{C}(A, B)$ , take a diagram

$$\begin{array}{ccc} L_f & \xrightarrow{\ell_f} & A \\ & & \downarrow f \\ & & B \\ & \searrow v_B \circ f & \downarrow v_B \\ & & V_B[1] \end{array}$$

$W_B$   
 $\downarrow w_B$

where

$$\begin{aligned} V_B &\rightarrow W_B \xrightarrow{w_B} B \xrightarrow{v_B} V_B[1] \\ V_B &\rightarrow L_f \xrightarrow{\ell_f} A \xrightarrow{v_B \circ f} V_B[1] \end{aligned}$$

are distinguished triangles satisfying  $W_B \in \mathcal{W}, V_B \in \mathcal{V}$ .

Then, the following holds.

- (1)  $\underline{f} \circ \underline{\ell}_f = 0$ .
- (2)  $\ell_f$  induces a bijection

$$\underline{\ell}_f \circ -: \underline{\mathcal{C}}(X, L_f) \xrightarrow{\cong} \{\alpha \in \underline{\mathcal{C}}(X, A) \mid \underline{f} \circ \alpha = 0\}$$

for any  $X \in \mathcal{C}^-$ .

- (3) If  $A \in \mathcal{C}^+$ , then  $L_f \in \mathcal{C}^+$ .

**Corollary 4.4.** *For any twin cotorsion pair,  $\underline{\mathcal{H}}$  is preabelian.*

*Proof.* First we construct a cokernel. For any  $A, B \in \mathcal{H}$  and any  $f \in \mathcal{C}(A, B)$ , let  $m_f: B \rightarrow M_f$  be as in Definition 4.1. Since  $A, B \in \mathcal{C}^-$ , it follows

$$\underline{m}_f \circ \underline{f} = 0, \quad M_f \in \mathcal{C}^-$$

by Proposition 4.2. By Proposition 3.6, there exists  $z_{M_f}: M_f \rightarrow Z_{M_f}$  which gives a reflection  $\underline{z}_{M_f}: M_f \rightarrow Z_{M_f}$  of  $M_f$  along  $\underline{\mathcal{C}}^+ \hookrightarrow \underline{\mathcal{C}}$ . By Claim 3.4,  $Z_{M_f}$  satisfies  $Z_{M_f} \in \mathcal{H}$ .

Then  $\underline{z}_{M_f} \circ \underline{m}_f: B \rightarrow Z_{M_f}$  gives a cokernel of  $\underline{f}$ . In fact for any  $H \in \mathcal{H}$ , there is a bijection

$$- \circ \underline{z}_{M_f} \circ \underline{m}_f : \underline{\mathcal{C}}(Z_{M_f}, H) \xrightarrow{\cong} \underline{\mathcal{C}}(M_f, H) \xrightarrow{\cong} \{\beta \in \underline{\mathcal{C}}(B, H) \mid \beta \circ \underline{f} = 0\}.$$

$$\begin{array}{ccccc}
& & 0 & \xrightarrow{\quad} & H \\
& \nearrow & \circlearrowleft & \nearrow \beta & \\
A & \xrightarrow{\underline{f}} & B & & \\
& \searrow \underline{m}_f & & & \\
& & M_f & \xrightarrow{\underline{z}_{M_f}} & Z_{M_f}
\end{array}$$

A kernel of  $\underline{f} \in \underline{\mathcal{H}}(A, B)$  is constructed dually. Let  $L \xrightarrow{\ell_f} A$  be as in Remark 4.3, and let  $K_{L_f} \xrightarrow{k_{L_f}} L_f$  be as in Definition 3.1. Then  $\underline{\ell}_f \circ \underline{k}_{L_f}$  gives a kernel of  $\underline{f}$ .  $\square$

**Corollary 4.5.** *Let  $f \in \mathcal{H}(A, B)$  be a morphism in  $\mathcal{H}$ . The following are equivalent.*

- (1)  $\underline{f} \in \underline{\mathcal{H}}(A, B)$  is epimorphic.
- (2)  $\underline{Z}_{M_f} \in \mathcal{W}$ .
- (3)  $M_f \in \mathcal{U}$ .

*Proof.* (1) is equivalent to (2), since  $\text{cok} \underline{f} \cong Z_{M_f}$  in  $\underline{\mathcal{H}}$ . Also (2) is equivalent to (3) by Lemma 3.9.  $\square$

**Corollary 4.6.** *Let  $f \in \mathcal{H}(A, B)$  be a morphism in  $\mathcal{H}$ . If a distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \rightarrow A[1]$$

*admits a factorization*

$$\begin{array}{ccc}
B & \xrightarrow{g} & C \\
& \searrow b & \nearrow c \\
& & U
\end{array}$$

*for some  $U \in \mathcal{U}$ , then  $\underline{f} \in \underline{\mathcal{H}}(A, B)$  is epimorphic.*

*Proof.* By the definition of  $B \xrightarrow{m_f} M_f$ , there is a commutative diagram made of distinguished triangles as follows (Definition 4.1).

$$\begin{array}{ccccc}
S_A[-1] & & & & \\
\downarrow s_A & \searrow & & \nearrow g & \\
& & B & & C \\
& \nearrow f & \searrow \circlearrowleft & \nearrow d & \\
A & \xrightarrow{\quad} & M_f & & \\
\downarrow w_A & \searrow e & & & \\
& & W_A & & 
\end{array}$$

By Corollary 4.5, it suffices to show  $M_f \in \mathcal{U}$ . Take any  $V^\dagger \in \mathcal{V}$  and  $v \in \mathcal{C}(M_f, V^\dagger[1])$ . Since  $v \circ e = 0$  by  $\text{Ext}^1(W_A, V^\dagger) = 0$ , there exists  $v' \in \mathcal{C}(C, V^\dagger[1])$

satisfying  $v' \circ d = v$ .

$$\begin{array}{ccccc}
 & & B & & \\
 & m_f \swarrow & & \searrow g & \\
 W_A & \xrightarrow{e} & M_f & \xrightarrow{d} & C \\
 & \searrow v & & \swarrow v' & \\
 & & V^\dagger[1] & & 
 \end{array}$$

Since  $g$  factors through  $U \in \mathcal{U}$ , it follows  $v \circ m_f = v' \circ d \circ m_f = v' \circ g = 0$ . Thus  $v$  factors through  $S_A$ ,

$$\begin{array}{ccccc}
 S_A[-1] & \xrightarrow{f \circ s_A} & B & \xrightarrow{m_f} & M_f & \longrightarrow & S_A \\
 & & \searrow \circ & & \searrow \circ & & \\
 & & & & v & & \\
 & & & & \searrow & & \\
 & & & & V^\dagger[1] & & 
 \end{array}$$

which means  $v = 0$ , since  $\text{Ext}^1(S_A, V^\dagger) = 0$ .  $\square$

*Remark 4.7.* Duals of Corollary 4.5 and 4.6 also hold for monomorphisms in  $\underline{\mathcal{H}}$ .

## 5. $\underline{\mathcal{H}}$ IS SEMI-ABELIAN

**Lemma 5.1.** *Let  $\beta \in \underline{\mathcal{H}}(B, C)$  be any morphism. If  $\beta$  is a cokernel morphism, namely, if there exists a morphism  $f \in \mathcal{H}(A, B)$  such that  $\beta = \text{cok} \underline{f}$ , then there exist  $g \in \mathcal{H}(B, C')$  and an isomorphism  $\eta \in \underline{\mathcal{H}}(C, C')$  such that*

(i)  $\eta$  is compatible with  $\beta$  and  $\underline{g}$ ,

$$\begin{array}{ccc}
 B & \xrightarrow{\beta} & C \\
 \searrow \underline{g} & \circ \eta & \searrow \cong \\
 & C' & 
 \end{array}$$

(ii)  $g$  admits a distinguished triangle

$$S[-1] \xrightarrow{s} B \xrightarrow{g} C' \rightarrow S$$

with  $S \in \mathcal{S}$ .

*Proof.* Take a morphism  $f \in \mathcal{H}(A, B)$  such that  $\beta = \text{cok} \underline{f}$ . As shown in Corollary 4.4,  $\text{cok} \underline{f}$  is given by  $\underline{z}_{M_f} \circ \underline{m}_f$ .

$$\begin{array}{ccc}
 S_A[-1] & & S[-1] \\
 \downarrow s_A & \searrow \circ & \downarrow \circ \\
 A & \xrightarrow{f} & B & \xrightarrow{m_f} & M_f \\
 \downarrow v_A & & & & \downarrow \circ \\
 W_A & & & & T \\
 & & & & \uparrow \circ \\
 & & & & U \\
 & & & & \uparrow \circ \\
 & & & & M_f \\
 & & & & \uparrow \circ \\
 & & & & Z_{M_f} \\
 & & & & \uparrow \circ \\
 & & & & V[1]
 \end{array}$$

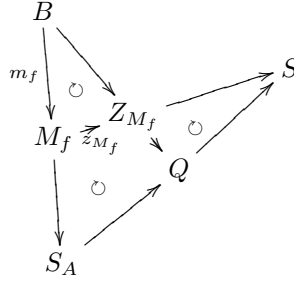
Thus there exists an isomorphism  $\eta \in \underline{\mathcal{H}}(C, Z_{M_f})$  compatible with  $\underline{z}_{M_f} \circ \underline{m}_f$  and  $\beta$ . It suffices to show  $g = z_{M_f} \circ m_f$  satisfies condition (ii). If we complete  $g$  into a distinguished triangle

$$Q[-1] \rightarrow B \xrightarrow{z_{M_f} \circ m_f} Z_{M_f} \rightarrow Q,$$

then by the octahedral axiom, we have a distinguished triangle

$$S_A \rightarrow Q \rightarrow S \rightarrow S_A[1],$$

which implies  $Q \in \mathcal{S}$ .



□

**Lemma 5.2.** *Suppose  $X \in \mathcal{C}^-$ ,  $B \in \mathcal{H}$  and  $x \in \mathcal{C}(X, B)$  admit a distinguished triangle*

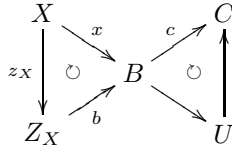
$$X \xrightarrow{x} B \rightarrow U \rightarrow X[1]$$

*with some  $U \in \mathcal{U}$ . Then, the unique morphism  $\zeta \in \underline{\mathcal{H}}(Z_X, B)$  satisfying  $\zeta \circ \underline{z}_X = \underline{x}$  (Proposition 3.6) becomes epimorphic.*

*Proof.* As in Proposition 3.6 (or, dual of the proof of Proposition 3.5), we see that there exists  $b \in \mathcal{C}(Z_X, B)$  satisfying  $\zeta = \underline{b}$  and  $b \circ z_X = x$ . If we complete  $b$  into a distinguished triangle

$$C[-1] \rightarrow Z_X \xrightarrow{b} B \xrightarrow{c} C,$$

then  $c$  factors through  $U$ .



Thus Lemma 5.2 follows from Corollary 4.6. □

**Lemma 5.3.** *Let*

$$(5.1) \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta \downarrow & \square & \downarrow \gamma \\ C & \xrightarrow{\delta} & D \end{array}$$

*be a pullback diagram in  $\underline{\mathcal{H}}$ . If there exist  $X \in \mathcal{C}^-$ ,  $x_B \in \mathcal{C}(X, B)$ ,  $x_C \in \mathcal{C}(X, C)$  which satisfies the following conditions, then  $\alpha$  is epimorphic.*

(i) *The following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{x_B} & B \\ \underline{x}_C \downarrow & \circlearrowleft & \downarrow \gamma \\ C & \xrightarrow{\delta} & D \end{array}$$

(ii) *There exists a distinguished triangle  $X \xrightarrow{x_B} B \rightarrow U \rightarrow X[1]$  with  $U \in \mathcal{U}$ .*

*Proof.* Take  $X \xrightarrow{z_X} Z_X$  as in Definition 3.3. By the adjointness, there exist  $\zeta_B \in \underline{\mathcal{H}}(Z_X, B)$  and  $\zeta_C \in \underline{\mathcal{H}}(Z_X, C)$  satisfying

$$\zeta_B \circ z_X = x_B, \quad \zeta_C \circ z_X = \underline{x}_C.$$

By Lemma 5.2,  $\zeta_B$  is epimorphic. From  $\gamma \circ x_B = \delta \circ \underline{x}_C$ , it follows  $\gamma \circ \zeta_B = \delta \circ \zeta_C$ .

$$\begin{array}{ccc} X & \xrightarrow{x_B} & B \\ \searrow z_X \circlearrowleft & & \downarrow \gamma \\ & Z_X & \xrightarrow{\zeta_B} B \\ \downarrow \underline{x}_C \circlearrowleft & \swarrow \zeta_C \circlearrowleft & \\ C & \xrightarrow{\delta} & D \end{array}$$

Since (5.1) is a pullback diagram in  $\underline{\mathcal{H}}$ , there exists  $\zeta \in \underline{\mathcal{H}}(Z_X, A)$  which satisfies  $\alpha \circ \zeta = \zeta_B$  and  $\beta \circ \zeta = \zeta_C$ .

$$\begin{array}{ccccc} Z_X & \xrightarrow{\zeta_B} & B & & \\ \searrow \zeta \circlearrowleft & & \downarrow \gamma & & \\ & A & \xrightarrow{\alpha} & B & \\ \downarrow \underline{x}_C \circlearrowleft & \downarrow \beta & \square & \downarrow \gamma & \\ C & \xrightarrow{\delta} & D & & \end{array}$$

Since  $\zeta_B$  is epimorphic,  $\alpha$  is also an epimorphism. □

**Theorem 5.4.** *For any twin cotorsion pair,  $\underline{\mathcal{H}}$  is semi-abelian.*

*Proof.* By duality, we only show  $\underline{\mathcal{H}}$  is left semi-abelian. Assume we are given a pullback diagram

$$(5.2) \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta \downarrow & \square & \downarrow \gamma \\ C & \xrightarrow{\delta} & D \end{array}$$

in  $\underline{\mathcal{H}}$ , where  $\delta$  is a cokernel morphism. It suffices to show  $\alpha$  becomes epimorphic.

By Lemma 5.1, replacing  $D$  by an isomorphic one if necessary, we may assume there exists  $d \in \mathcal{H}(C, D)$  satisfying  $\delta = \underline{d}$ , which admits a distinguished triangle

$$S[-1] \rightarrow C \xrightarrow{d} D \xrightarrow{s} S$$

with  $S \in \mathcal{S}$ . If we take  $c \in \mathcal{H}(B, D)$  satisfying  $\gamma = \underline{c}$ , and complete  $s \circ c$  into a distinguished triangle

$$S[-1] \rightarrow X \xrightarrow{x_B} B \xrightarrow{s \circ c} S,$$

then  $c \circ x_B$  factors through  $d$ . In fact, there exists  $x_C \in \mathcal{C}(X, C)$  which gives a morphism of triangles as follows.

$$\begin{array}{ccccccc} S[-1] & \longrightarrow & X & \xrightarrow{x_B} & B & \xrightarrow{s \circ c} & S \\ \parallel & & \circlearrowleft & x_C \downarrow & \circlearrowleft & \downarrow c & \circlearrowleft \\ S[-1] & \longrightarrow & C & \xrightarrow{d} & D & \xrightarrow{s} & S \end{array}$$

By Lemma 2.12, we have  $X \in \mathcal{C}^-$ . Thus  $\alpha$  becomes epimorphic by Lemma 5.3.  $\square$

## 6. THE CASE WHERE $\underline{\mathcal{H}}$ BECOMES INTEGRAL

In the rest, additionally we assume that  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  satisfies

$$(6.1) \quad \mathcal{U} \subseteq \mathcal{S} * \mathcal{T} \quad \text{or} \quad \mathcal{T} \subseteq \mathcal{U} * \mathcal{V}.$$

This condition is satisfied, for example in the following cases.

**Example 6.1.** A twin cotorsion pair  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  satisfies (6.1) in the following cases.

- (1)  $\mathcal{U} = \mathcal{S}$ . Namely,  $(\mathcal{S}, \mathcal{T}) = (\mathcal{U}, \mathcal{V})$  is a single cotorsion pair.
- (2)  $\mathcal{U} = \mathcal{T}$ . For example, Buan and Marsh's triplet  $(\text{add}(T)[1], \mathcal{X}_T, (\mathcal{X}_T)^\perp[-1])$ .
- (3)  $(\mathcal{S}, \mathcal{T})$  is a co- $t$ -structure. In this case,  $\mathcal{S} * \mathcal{T} = \mathcal{C}$ .
- (4)  $(\mathcal{U}, \mathcal{V})$  is a co- $t$ -structure. In this case,  $\mathcal{U} * \mathcal{V} = \mathcal{C}$ .

Remark that the following holds.

**Fact 6.2.** ([R]) A semi-abelian category  $\mathcal{A}$  is left integral if and only if  $\mathcal{A}$  is right integral.

**Theorem 6.3.** *If a twin cotorsion pair  $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$  satisfies (6.1), then  $\underline{\mathcal{H}}$  becomes integral.*

*Proof.* By duality, it suffices to show that  $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$  implies left integrality.

Let  $b \in \mathcal{H}(B, D)$  and  $c \in \mathcal{H}(C, D)$  be morphisms satisfying  $\beta = \underline{b}$  and  $\delta = \underline{d}$ . Since  $\delta$  is epimorphic, if we take  $D \xrightarrow{m_d} M_d$  as in Definition 4.1, then  $M_d \in \mathcal{U}$  by Corollary 4.5.

$$\begin{array}{ccc} S_C[-1] & & \\ s_C \downarrow & \searrow & \\ C & \xrightarrow{d} & D \\ w_C \downarrow & & \searrow m_d \\ W_C & & M_d \end{array}$$

By assumption  $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ , there exists a distinguished triangle

$$S_0 \xrightarrow{s_0} M_d \xrightarrow{t_0} T_0 \rightarrow S_0[1]$$

with  $S_0 \in \mathcal{S}, T_0 \in \mathcal{T}$ .

If we take a distinguished triangle

$$S_B[-1] \xrightarrow{s_B} B \xrightarrow{w_B} W_B \rightarrow S_B \quad (S_B \in \mathcal{S}, W_B \in \mathcal{W}),$$



then by  $\text{Ext}^1(S_B, T_0) = 0$ ,  $m_d \circ c \circ s_B$  factors through  $s_0$ . Namely, there exists  $g \in \mathcal{C}(S_B[-1], S_0)$  which makes the following diagram commutative.

$$\begin{array}{ccc} S_B[-1] & \xrightarrow{g} & S_0 \\ s_B \downarrow & \circlearrowleft & \downarrow s_0 \\ B & & \\ c \downarrow & & \downarrow \\ D & \xrightarrow{m_d} & M_d \end{array}$$

If we complete  $g$  into a distinguished triangle

$$S_0[-1] \rightarrow X \xrightarrow{f_B} S_B[-1] \xrightarrow{g} S_0,$$

then  $X \in \mathcal{S}[-1] \subseteq \mathcal{C}^-$ . Moreover there exists  $f_C \in \mathcal{C}(X, S_C[-1])$  satisfying  $d \circ s_C \circ f_C = c \circ s_B \circ f_B$ .

$$\begin{array}{ccccccc} S_0[-1] & \longrightarrow & X & \xrightarrow{f_B} & S_B[-1] & \xrightarrow{g} & S_0 \\ \downarrow & \circlearrowleft & \downarrow f_C & \circlearrowleft & \downarrow c \circ s_B & \circlearrowleft & \downarrow s_B \\ M_d[-1] & \rightarrow & S_C[-1] & \xrightarrow{d \circ s_C} & D & \xrightarrow{m_d} & M_d \end{array}$$

Thus we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s_B \circ f_B} & B \\ s_C \circ f_C \downarrow & \circlearrowleft & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

If we complete  $s_B \circ f_B$  into a distinguished triangle

$$X \xrightarrow{s_B \circ f_B} B \rightarrow Q \rightarrow X[1],$$

then by the octahedral axiom, we have  $Q \in \mathcal{U}$ . Thus by Lemma 5.3,  $\alpha$  becomes epimorphic.

$$\begin{array}{ccccc} X & & & & \\ \downarrow f_B & \searrow & & \nearrow & \\ S_B[-1] & \xrightarrow{s_B} & B & \xrightarrow{w_B} & Q \\ & \searrow & \downarrow & \nearrow & \\ & & S_0 & & \end{array}$$

□

## REFERENCES

- [AN] Abe, N.; Nakaoka, H.: *General heart construction on a triangulated category (II): associated homological functor*, Appl. Categ. Structures, Online First 2010, DOI: 10.1007/s10485-010-9226-z.
- [BBD] Beilinson, A. A.; Bernstein, J.; Deligne, P.: *Faisceaux pervers* (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.

- [BR] Beligiannis, A.; Reiten, I.: *Homological and homotopical aspects of torsion theories* (English summary), Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207 pp.
- [B] Borceux, F.: *Handbook of categorical algebra 1, Basic category theory*, Encyclopedia of Mathematics and its Applications, **50**. Cambridge University Press, Cambridge, 1994. xvi+345 pp.
- [BM] Buan, A. B.; Marsh, R. J.: *From triangulated categories to module categories via localization II: calculus of fractions*, arXiv: 1011.4597.
- [BMR] Buan, A. B.; Marsh, R. J.; Reiten, I.: *Cluster-tilted algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 323–332.
- [IY] Iyama, O.; Yoshino, Y.: *Mutation in triangulated categories and rigid Cohen-Macaulay modules* (English summary), Invent. Math. **172** (2008), no. 1, 117–168.
- [KR] Keller, B.; Reiten, I.: *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.
- [KZ] Koenig, S.; Zhu, B.: *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. **258** (2008), no. 1, 143–160.
- [N] Nakaoka, H.: *General heart construction on a triangulated category (I): unifying  $t$ -structures and cluster tilting subcategories*, Appl. Categ. Structures, Online First 2010, DOI: 10.1007/s10485-010-9223-2.
- [R] Rump, W.: *Almost abelian categories*, Cahiers Topologie Géom. Différentielle. Catég. **42** (2003), no. 3, 163–225.
- [ZZ] Zhou, Y.; Zhu, B.: *Mutation of torsion pairs in triangulated categories and its geometric realization*, arXiv: 1105.3521.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KAGOSHIMA UNIVERSITY, 1-21-35  
KORIMOTO, KAGOSHIMA, 890-0065 JAPAN

*E-mail address:* nakaoka@sci.kagoshima-u.ac.jp